3.2: Quadratic Functions

Standard Form of Quadratic Functions. (Completing the square)

- If $f(x) = ax^2 + bx + c$ then the standard form of the function is $f(x) = a(x-h)^2 + k$.
- To find h and k,

$$
\begin{aligned}
ax^2 + bx + c &= a(x-h)^2 + k \\
ax^2 + bx + c &= a(x^2 - 2xh + h^2) + k \\
ax^2 + bx + c &= ax^2 \underbrace{-2ah}_{b}x + \underbrace{(ah^2 + k)}_{c}
\end{aligned}
$$

$\leftarrow$ Set the coefficients of the same terms equal.

$h = -\dfrac{b}{2a}$ and $k = c - \dfrac{b^2}{4a}$ Note: Memorize $h = -\dfrac{b}{2a}$ and $k = f(h)$ (Don't memorize $k = c - \dfrac{b^2}{4a}$).

Derivation of Quadratic Formula (Fun Fact)

- Quadratic formula is derived using the standard form.

$$\Longrightarrow a(x-h)^2 + k = 0$$

$$\Longrightarrow a(x-h)^2 = -k$$

$$\Longrightarrow (x-h)^2 = -\frac{k}{a}$$

$$\Longrightarrow x - h = \pm\sqrt{-\frac{k}{a}}$$

$$\Longrightarrow x = h \pm \sqrt{-\frac{k}{a}}$$

Replace h and k $\Longrightarrow$

$$x = -\frac{b}{2a} \pm \sqrt{-\frac{c - \frac{b^2}{4a}}{a}}$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{-4ac + b^2}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

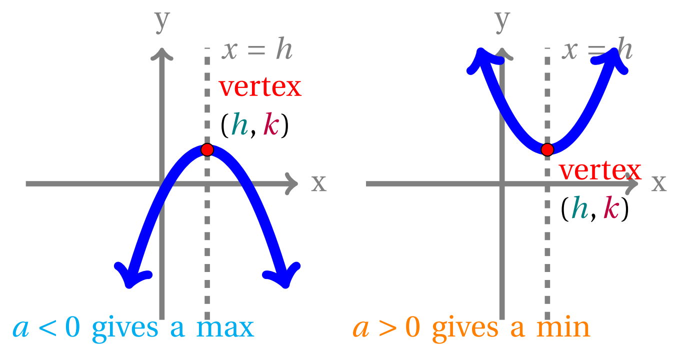
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The Graph of Quadratic Functions

- We call the point (h, k) the vertex of parabola. If $a > 0$ the parabola is upward. If $a < 0$ the parabola is downward.
- The maximum or minimum value of f is attained at the vertex.

  If $a > 0$ the minimum is at (h, k).

  If $a < 0$ the maximum is at (h, k).



- Use the standard form and transformation to graph any quadratic from parent function $y = x^2$.
- $x = h$ is the line of symmetry for the graph.

How to Model Optimization Problems

- Find the input and output variables. Output is the variable that is being optimized.
- Find constraints that relate all other variables to the input. Then write all other variables as a function of the input.
- Write the output as an expression of all other variables and then replace all by functions of input. This makes the output a function of input only.
- For quadratic functions: Use the formula $x = -\dfrac{b}{2a}$ to find the input that optimizes the output. Use $f\left(-\dfrac{b}{2a}\right)$ to find the optimized value.

Note: The following set of optimization problems follow the above steps. In precalculus, we get to optimize using a limited set of functions; quadratic functions belong to this set. A much larger set of function can be optimized with calculus. Other functions will be discussed later.

1. Find the maximum or minimum values of the following functions. At what value of x does the max or min occur?

   (a) $f(x) = -x^2 + 6x - 5$

   (b) $g(x) = 100x^2 - 2800x$

2. Find a parabola whose vertex is $(1, -3)$ and passes through $(4, 16)$.

3. **Physics:** A ball is thrown across a playing field. The path of the ball is modeled by the function

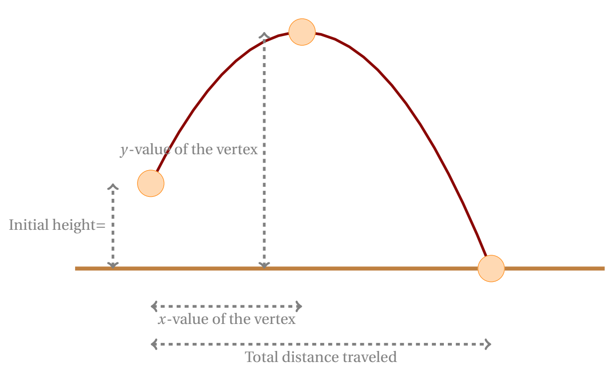
$$y = -x^2 + 4x + 2.25$$

   where x is the distance in meters that the ball has traveled horizontally and y is the height of the ball in meters when the ball has traveled x meters.

   (A) What is the initial height of the ball when it is thrown?

   (B) **An Optimization Problem:** Find the maximum height attained by the ball in meters.

   (C) Find the horizontal distance the ball has traveled when it hits the ground.



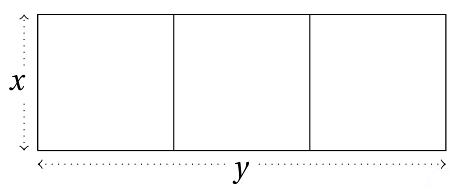
4. **Business and Econ:** Lyle's Lemonade Stand sells glasses of lemonade at baseball games for \$1 a glass. At that price he sells 300 glasses of lemonade a game. He's noticed that for every 5 cents He raises the price, he sells 10 fewer glasses. Let x be the **number of times** Lyle **raises** the price by 5 cents.

   (a) What is the number of glasses sold in terms of a function of x? What is the price of each glass in terms of a function of x?

   (b) Express the revenue as a function of x.

   (c) **An Optimization Problem:** What is the **maximum revenue**? At what **price** is the maximum revenue is earned?

5. Yiying is constructing a garden which will be separated into 3 plots as shown, where x and y are the width and the length of the garden in yards:



   Yiying will surround the garden by a rectangular fence, and separate the plots with fencing material. She has 64 yards of fencing material to use.

   (a) Express the dimension y as a function of x.

   (b) Find a function that models the **area** of the garden as a function of x.

   (c) **An Optimization Problem:** What are the dimensions, x and y, that will maximize the area of the garden? And what is the maximum area of the garden?

6. **An Optimization Problem:**Yanru has 1200 feet of fencing to enclose a rectangular plot of land. Two sides of the rectangle will have length x and the other two will have length $600 - x$. What is the maximum area the farmer can enclose? *(Circle only one)*

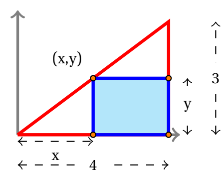
(A) $300ft^2$

(B) $1,500ft^2$

(C) $45,000ft^2$

(D) $90,000ft^2$

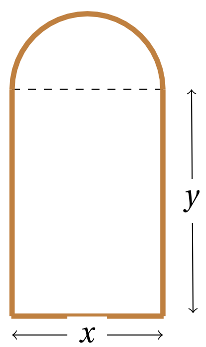
7. **An Optimization Problem:** What is the **maximum area** of a rectangle inscribed in a right triangle with side lengths 4 and 3, if the sides of the rectangle are parallel to the legs of the triangle?



Click here to watch the area https://ggbm.at/qf4svwwc.

8. **An Average Rate of Change Problem:** Let $f(x) = x^2 - 3x - 10$. Find average rate of change in f on interval $[a, a+h]$ and simplify. (This is also called the **difference quotient** for $f(x)$.)

9. **An Optimization Problem:** A Norman window has the shape of a rectangle surmounted by a semi-circle. Find the dimensions of a Norman window of perimeter 30 ft that will admit the greatest possible amount of light.



10. **An Optimization Problem:** A piece of wire 10 cm long is bent into a rectangle. What dimensions produce the rectangle with maximum area?

    Watch the area here: https://ggbm.at/xu4th4jg

Example Videos:

1. https://mediahub.ku.edu/media/t/1_aln2hs1o
2. https://mediahub.ku.edu/media/t/1_3j2xt1iu